

Refined Probability of Differential Characteristics Including Dependency Between Multiple Rounds

Anne Canteaut¹, Eran Lamboojij², Samuel Neves³, Shahram Rasoolzadeh⁴,
Yu Sasaki⁵ and Marc Stevens⁶

¹ Inria

² Technische Universiteit Eindhoven

³ CISUC, Dept. of Informatics Engineering, University of Coimbra

⁴ Ruhr-Universität Bochum

⁵ NTT Secure Platform Laboratories

⁶ CWI Amsterdam

Abstract. The current paper studies the probability of differential characteristics for an unkeyed (or with a fixed key) construction. Most notably, it focuses on the gap between two probabilities of differential characteristics: probability with independent S-box assumption, p_{ind} , and exact probability, p_{exact} . It turns out that p_{exact} is larger than p_{ind} in Feistel network with some S-box based inner function. The mechanism of this gap is then theoretically analyzed. The gap is derived from interaction of S-boxes in three rounds, and the gap depends on the size and choice of the S-box. In particular the gap can never be zero when the S-box is bigger than six bits. To demonstrate the power of this improvement, a related-key differential characteristic is proposed against a lightweight block cipher ROADRUNNER. For the 128-bit key version, p_{ind} of 2^{-48} is improved to p_{exact} of 2^{-43} . For the 80-bit key version, p_{ind} of 2^{-68} is improved to p_{exact} of 2^{-62} . The analysis is further extended to SPN with an almost-MDS binary matrix in the core primitive of the authenticated encryption scheme Minalpher: p_{ind} of 2^{-128} is improved to p_{exact} of 2^{-96} , which allows to extend the attack by two rounds.

Keywords: differential cryptanalysis · independent S-box · fixed key · unkeyed construction · exact probability · ROADRUNNER · Minalpher

1 Introduction

Differential cryptanalysis [BS90, BS93] is one of the most fundamental cryptanalytic approaches targeting symmetric-key primitives. While its basic concept in an idealized environment under several assumptions can easily be understood, predicting the actual behavior of concrete algorithms is quite complex and a lot of research has been done regarding this topic.

Most block ciphers are designed to iterate a small keyed permutation, called the round function, with many rounds being performed to build a conversion between the plaintext and ciphertext. The plaintext x_0 is updated by round function RF_i in the i th round by processing $x_{i+1} \leftarrow RF_i(x_i)$ for $i = 0, 1, 2, \dots$. The most common approach for evaluating the effect of differential analysis consists in applying the Markov assumption to the cipher [LMM91] and evaluating the probability of differential propagation for each round. The probability of the differential characteristic over the entire cipher is then equal to the product of the probabilities of the differentials of all rounds.

Given a pair of differences (a_i, a_{i+1}) and the corresponding probability p_i such that $p_i \triangleq \Pr_{x \in \mathbf{P}}[RF_i(x) \oplus RF_i(x \oplus a_i) = a_{i+1}]$ is searched for each i , where \mathbf{P} is the plaintext

space, and $\Pi_i p_i$ is the probability of the characteristic (a_0, a_1, \dots, a_r) for the entire r -round cipher.

The hidden argument in the above explanation is the treatment of a key k or subkeys k_i . The Markov assumption can be established when the state x_i is first xored with a subkey k_i and all subkeys are chosen independently uniformly at random. Therefore, most analyses are based on bounds on the *expected* probability of a differential characteristic, i.e., the probability averaged over all keys. However, implementation environment for symmetric-key primitives does not allow to store all independent subkeys, thus k_i is usually expanded from k , and the Markov assumption collapses.

Moreover, subkeys may not be xored in every round to all state bits, which can be seen in designs of lightweight cryptographic schemes such as SIMON [BSS⁺13], SKINNY [BJK⁺16] and LED [GPPR11]. Also some primitives, like hash functions or Even-Mansour schemes [DKS12, EM91, EM97], are based on an iterated permutation which does not involve any key at all. In such a case, the evaluation using the Markov assumption may still give some insight about the security against differential analysis, but never leads to the exact probability of the differential propagation for multiple rounds.

To conclude, evaluating the probability of differential propagations for multiple rounds precisely without the Markov assumption is a big challenge.

1.1 Related Work on Precise Evaluation of Differential Probability

Our work then focuses on the evaluation of the probability of a differential characteristic for a primitive with a fixed key, or for a keyless primitive. It is worth noticing that both contexts are equivalent. Indeed, the absence of a key can equivalently be seen as the insertion of an all-zero key. Conversely, a structure with a fixed key is equivalent to an unkeyed one with different building blocks. For instance, using an S-box S with a fixed round-key k is equivalent to using $S' : x \mapsto S_k(x)$ as an S-box without any key. Let E be a block cipher with a fixed key and let ΔP and ΔC be the plaintext and ciphertext differences, respectively. Suppose that the goal is to precisely evaluate the probability of $\Pr[E(x) \oplus E(x \oplus \Delta P) = \Delta C]$, where the probability is taken over all plaintexts x . Besides the issue of subkeys for multiple rounds, there are several aspects to precisely evaluate this probability.

The first issue we would like to mention is the contrast between differential characteristics and differential effect. The differential characteristics specify not only $(\Delta P, \Delta C)$ but also differences in intermediate states, often the initial difference in each round, and evaluate the probability of each section and multiplies all the probabilities. On the contrary, the differential effect sums up the probabilities of all possible differential characteristics, thus gives a more precise probability. A lot of research has been done to evaluate the exact maximum expected differential probability in particular for AES, e.g. [HLL⁺00, KMT01, PSC⁺02, PSL03, DR06, KS07, CR15]. Those researches are different from the current paper with respect to the point that all state bits are xored by subkeys which are assumed to be chosen independently and uniformly at random.

In contrary, our work focuses on determining the exact probability of a differential characteristic when the key is fixed. This fixed-key probability has been determined in a very few cases only. The most prominent example is the AES, for which the probability of 2-round characteristics has been determined, for all possible values of the key [DR07].

Alternative approaches can be used when such a theoretical analysis is out of reach. One approach is carrying out some experiment, which exhaustively chooses plaintexts $P \in \mathbf{P}$ and actually computes $E_K(x) \oplus E_K(x \oplus \Delta P)$. The experiment is then iterated for several keys (see e.g. [BG10]). The experiment can include any complex event, however, the lack of theoretical analysis limits its versatility to be applied to other ciphers. Of course the approach can only be applied to ciphers with small block sizes, often 32-bit block sizes, such as SIMON and KATAN [DDGS15, CDK09]. Another approach introduced

in [BBL13] consists in computing the maximal expected probability of a characteristic and deriving a bound on the probability of the existence of characteristics whose fixed-key probability exceeds a given value. This result can be used by designers to guarantee that characteristics with high probability are very unlikely. However, this bound exhibits a large gap between the fixed-key and the expected probabilities (see Table 1 in [BBL13]). It is then of little use to the cryptanalyst who needs to estimate the exact probability of some characteristic for a given key.

1.2 Our Contributions

In this paper, we evaluate the exact probabilities of the differential characteristics in some unkeyed constructions. In particular, we provide an in-depth study of the probabilities of the differential characteristics over three rounds of an unkeyed Feistel network. Most notably, when the inner function follows an SPN construction with an S-box having differential uniformity 4, the exact probability of a 3-round characteristic is either zero or a value which is greater than or equal to the usual estimate with independent S-box assumption, p_{ind} . A more thorough analysis is then provided when the inner function consists of a single n -bit S-box with differential uniformity 4. We show that, in this case, the exact probability of any 3-round characteristic is either zero, or exceeds p_{ind} by a factor of 2^ℓ where $\ell \geq \max(0, n - 6)$.

The above analysis is then applied to the lightweight 64-bit block cipher ROADRUNNER [BS15]. It adopts a Feistel network and its inner function starts and ends with the S-box application without applying any subkey, therefore the above generic analysis can be applied. Although no security is claimed against related-key attacks, the designers mentioned related-key differential characteristics with 24 active S-boxes on the full (12) rounds of ROADRUNNER-128, whose probability is expected to be $2^{-2 \cdot 24} = 2^{-48}$. The designers also speculated that the number of active S-boxes could be reduced further with more careful analysis. In this paper, we first concretize the related-key characteristic with 24 active S-boxes and show that the exact probability is higher than the original expectation. The comparison of two probabilities is shown in Table 1. The attack is implemented up to 8 rounds and the improved factor is verified. We prove that the minimum number of active S-boxes is 24 by using a SAT solver, thus our characteristic is fairly tight.

Finding related-key differential characteristics is much harder in ROADRUNNER-80 due to its key schedule. We propose an 8-round characteristic with $p_{\text{ind}} = 2^{-68}$ which are unlikely to be satisfied even with a full codebook, but the improvement with p_{exact} increases it to 2^{-62} .

Table 1: Improved probability of characteristics for ROADRUNNER-128 and Minalpher.

Rounds	1	2	3	4	5	6	7	8	9	10	11	12
ROADRUNNER-128												
p_{ind}	-4	-8	-12	-16	-20	-24	-28	-32	-36	-40	-44	-48
p_{exact}	-4^\dagger	-8^\dagger	-12^\dagger	-15^\dagger	-19^\dagger	-22^\dagger	-26^\dagger	-29^\dagger	-33	-36	-40	-43
ROADRUNNER-80												
p_{ind}	-8	-17	-26	-34	-42	-51	-60	-68				
p_{exact}	-8^\dagger	-17^\dagger	-25^\dagger	-32^\dagger	-39^\dagger	-47	-55	-62				
Minalpher												
p_{ind}	-16	-48	-64	-80	-112	-128						
p_{exact}	-16^\dagger	-40^\ddagger	-48	-64	-88	-96	-112	-128				

Numbers denote logarithm of the probabilities. Probabilities with † were experimentally verified. Probability with ‡ was experimentally verified only for the essential part.

We then extend the application of our observations to SPN-based structures with almost-

MDS binary matrices. In particular, we analyze p_{exact} of the differential characteristic in an authenticated encryption scheme Minalpher [STA⁺14], which offers 128-bit security. The previous differential characteristic reaches 2^{-128} for 6 (out of 17.5) rounds. We show that for this characteristic the exact probability is 2^{-96} . This significant increase enables us to extend the attack by two rounds. The comparison of the probabilities are given in Table 1.

1.3 Paper Outline

The paper is organized as follows. Section 2 provides theoretical analysis of p_{exact} for 3-round Feistel structure. Section 3 applies the observation to ROADRUNNER with 128-bit key. Section 4 extends the application to SPN with almost-MDS matrices in Minalpher.

2 Probabilities of 3-Round Characteristics in some Keyless Feistel Networks

In this section, we evaluate the exact probability of a differential characteristic over three rounds of an unkeyed Feistel network whose inner function is seen as a single S-box application. As previously noted, this keyless scenario also covers the study of the same construction with fixed round-keys. We then want to determine the probability over all possible inputs (x_0, x_1) of the three-round characteristic depicted on Figure 1, where $\Delta_{i+1} = \Delta'_i \oplus \Delta_{i-1}$. It is worth noticing that the differential probabilities for an unkeyed 3-round Feistel have been previously investigated in order to determine the smallest differential uniformity we can get for an S-box which follows this construction [LW14, CDL15]. However, these papers focus on the maximum possible probability for a 3-round differential characteristic, while we want to obtain a formula which captures any given characteristic.

Using that $x_3 = S(x_2) \oplus x_1$, we get that the probability of the three-round characteristic defined by $(\Delta_0, \dots, \Delta_4)$ is equal to the following probability:

$$p_{\text{exact}} = \Pr_{x_1, x_2 \in \mathbb{F}_2^n} [S(S(x_2) \oplus x_1 \oplus \Delta'_2 \oplus \Delta_1) \oplus S(S(x_2) \oplus x_1) = \Delta'_3 \\ \text{and } S(x_2 \oplus \Delta_2) \oplus S(x_2) = \Delta'_2 \text{ and } S(x_1 \oplus \Delta_1) \oplus S(x_1) = \Delta'_1].$$

We will show that this probability may differ from the usual estimate obtained when assuming that the inputs of the three S-boxes are independent, i.e. from

$$p_{\text{ind}} = \Pr_{x_3 \in \mathbb{F}_2^n} [S(x_3 \oplus \Delta'_2 \oplus \Delta_1) \oplus S(x_3) = \Delta'_3] \times \Pr_{x_2 \in \mathbb{F}_2^n} [S(x_2 \oplus \Delta_2) \oplus S(x_2) = \Delta'_2] \\ \times \Pr_{x_1 \in \mathbb{F}_2^n} [S(x_1 \oplus \Delta_1) \oplus S(x_1) = \Delta'_1].$$

The difference between the two probabilities mainly comes from some dependencies due to the fact that the input of the S-box in the third round is the sum of two elements, x_1 and $S(x_2)$, where x_1 and x_2 respectively conform to the S-box differentials (Δ_1, Δ'_1) and (Δ_2, Δ'_2) . Also, we show that the size of the S-box and, for a given size, the choice of the S-box may affect the factor between the exact probability and the usual estimate.

More precisely, we first show that, in many cases, including when S has an SPN structure based on an S-box with differential uniformity at most 4, the factor λ between these two probabilities is a power of 2 whose exponent corresponds to the dimension of a well-defined linear space. Most notably, if S corresponds to a single S-box with differential uniformity at most 4, then

$$p_{\text{exact}} = \lambda p_{\text{ind}}.$$

with $\lambda \in \{0, 2^\ell\}$, with $\max(0, n-6) \leq \ell \leq n-2$.

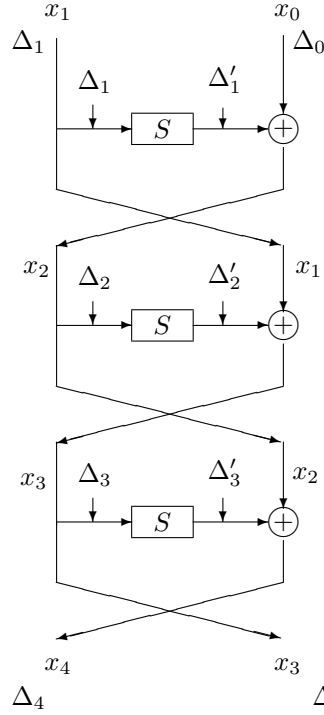


Figure 1: Differential characteristic of a three-round Feistel network where $\Delta_{i+1} = \Delta'_i \oplus \Delta_{i-1}$.

2.1 General result

The technique used in the proof is similar to the one used by Daemen and Rijmen for computing the fixed-key probabilities of the differentials over two rounds of the AES [DR07]. It mainly relies on the algebraic structure of the sets of inputs (resp. of outputs) of the S-box conforming to a given differential. These sets are defined as follows.

Definition 1. Let S be an n -bit to n -bit S-box. For any pair (a, b) of differences, we use the following notation:

$$\mathcal{X}_S(a, b) \triangleq \{x \in \mathbb{F}_2^n : S(x \oplus a) \oplus S(x) = b\},$$

and

$$\mathcal{Y}_S(a, b) \triangleq \{S(x) \in \mathbb{F}_2^n : S(x \oplus a) \oplus S(x) = b\}.$$

We focus on the following data transformation depicted on Figure 2:

$$z = S(x) \oplus y \text{ such that } x \in \mathcal{X}_S(a_1, b_1) \text{ and } y \in \mathcal{X}_S(a_2, b_2).$$

When the three sets $\mathcal{Y}_S(a_1, b_1)$, $\mathcal{X}_S(a_2, b_2)$ and $\mathcal{X}_S(b_1 \oplus a_2, b_3)$ are affine subspaces, we get the following result.

Proposition 1. Let S be a permutation of \mathbb{F}_2^n , and let a_1, b_1, a_2, b_2, b_3 be five nonzero elements in \mathbb{F}_2^n with $a_2 \neq b_1$. Assume that there exist $y_1, x_2, x_3 \in \mathbb{F}_2^n$ and three linear subspaces $V_1, V_2, V_3 \subseteq \mathbb{F}_2^n$ such that

$$\mathcal{Y}_S(a_1, b_1) = y_1 + V_1, \mathcal{X}_S(a_2, b_2) = x_2 + V_2, \text{ and } \mathcal{X}_S(b_1 \oplus a_2, b_3) = x_3 + V_3.$$

Then, the multiset

$$\{(x, y) \in \mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2) : S(S(x) \oplus y \oplus b_1 \oplus a_2) \oplus S(S(x) \oplus y) = b_3\}$$

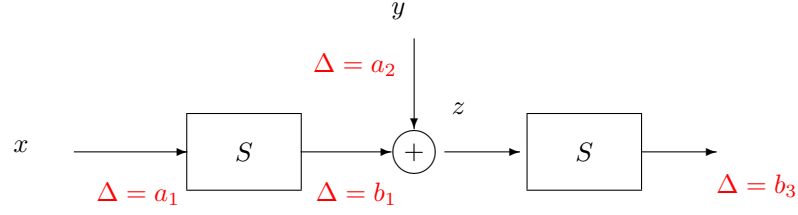


Figure 2: Target structure.

is either empty or has size 2^d with

$$d = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2 + V_3)$$

where $V_1 + V_2 + V_3$ denotes the linear space formed by all elements of the form $v_1 + v_2 + v_3$ with $v_i \in V_i$.

Proof. Let us define the following set (without multiplicity)

$$\mathcal{Z} = \{(S(x) \oplus y) : (x, y) \in \mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2)\}.$$

Then, $\mathcal{Z} = (y_1 \oplus x_2) + (V_1 + V_2)$, and each element in \mathcal{Z} corresponds to 2^r elements in $\mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2)$ with $r = \dim V_1 + \dim V_2 - \dim(V_1 + V_2)$. We want to determine the size of the set

$$\mathcal{S} = \{z \in \mathcal{Z} : S(z \oplus b_1 \oplus a_2) \oplus S(z) = b_3\}.$$

Clearly, this set corresponds to the intersection between \mathcal{Z} and $\mathcal{X}_S(b_1 \oplus a_2, b_3)$, which are both affine subspaces of \mathbb{F}_2^n . Since the intersection between two affine subspaces is either empty or a coset of the intersection between the corresponding linear subspaces, we deduce that, if $\mathcal{S} \neq \emptyset$, then there exists some s such that

$$\mathcal{S} = s + ((V_1 + V_2) \cap V_3).$$

It follows that, if $\mathcal{S} \neq \emptyset$, we have

$$\dim \mathcal{S} = \dim((V_1 + V_2) \cap V_3) = \dim(V_1 + V_2) + \dim V_3 - \dim(V_1 + V_2 + V_3).$$

Since each element in \mathcal{Z} and then in \mathcal{S} corresponds to 2^r pairs (x, y) in $\mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2)$, we deduce that the multiset

$$\{(x, y) \in \mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2) : S(S(x) \oplus y \oplus b_1 \oplus a_2) \oplus S(S(x) \oplus y) = b_3\}$$

is either empty or has size 2^d with

$$\begin{aligned} d &= r + \dim \mathcal{S} \\ &= \dim V_1 + \dim V_2 - \dim(V_1 + V_2) + \dim(V_1 + V_2) + \dim V_3 - \dim(V_1 + V_2 + V_3) \\ &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2 + V_3). \end{aligned}$$

□

Thus, we get the following corollary as a direct consequence of Proposition 1.

Corollary 1. *Let S be a permutation of \mathbb{F}_2^n , and let a_1, b_1, a_2, b_2, b_3 be five nonzero elements in \mathbb{F}_2^n with $a_2 \neq b_1$. Assume that there exist $y_1, x_2, x_3 \in \mathbb{F}_2^n$ and three linear subspaces $V_1, V_2, V_3 \subseteq \mathbb{F}_2^n$ such that*

$$\mathcal{Y}_S(a_1, b_1) = y_1 + V_1, \mathcal{X}_S(a_2, b_2) = x_2 + V_2, \text{ and } \mathcal{X}_S(b_1 \oplus a_2, b_3) = x_3 + V_3.$$

Let

$$p_{\text{exact}} = \Pr_{x, y \in \mathbb{F}_2^n} [S(S(x) \oplus y \oplus b_1 \oplus a_2) \oplus S(S(x) \oplus y) = b_3 \text{ and } S(x \oplus a_1) \oplus S(x) = b_1 \\ \text{and } S(y \oplus a_2) \oplus S(y) = b_2]$$

and

$$p_{\text{ind}} = \Pr_{x \in \mathbb{F}_2^n} [S(x \oplus a_1) \oplus S(x) = b_1] \times \Pr_{y \in \mathbb{F}_2^n} [S(y \oplus a_2) \oplus S(y) = b_2] \\ \times \Pr_{z \in \mathbb{F}_2^n} [S(z \oplus b_1 \oplus a_2) \oplus S(z) = b_3]$$

Then,

$$p_{\text{exact}} = 2^\ell p_{\text{ind}} \text{ with } \ell = n - \dim(V_1 + V_2 + V_3).$$

Most notably, $0 \leq \ell \leq n - 2$.

Proof. Let us focus on the case where $p_{\text{exact}} \neq 0$. We deduce from Proposition 1 that

$$p_{\text{exact}} = 2^{\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2 + V_3) - 2n}.$$

Since $p_{\text{ind}} = 2^{\dim V_1 + \dim V_2 + \dim V_3 - 3n}$, we deduce that

$$\lambda = \frac{p_{\text{exact}}}{p_{\text{ind}}} = 2^\ell$$

with

$$\ell = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2 + V_3) - 2n - (\dim V_1 + \dim V_2 + \dim V_3 - 3n) \\ = n - \dim(V_1 + V_2 + V_3).$$

Since $V_1 + V_2 + V_3$ is a subspace of \mathbb{F}_2^n , its dimension does not exceed n . On the other hand, when $p_{\text{ind}} \neq 0$, V_1 (resp. V_2) contains at least two elements, y and $y \oplus b_1$ (resp. x and $x \oplus a_2$). It follows that $V_1 + V_2$ contains $\langle b_1, a_2 \rangle$ which has dimension 2 because $b_1 \neq a_2$. We derive that

$$2 \leq \dim(V_1 + V_2 + V_3) \leq n$$

implying that

$$0 \leq \ell \leq n - 2.$$

□

The hypothesis required for applying by this result, i.e., the fact that the three sets $\mathcal{Y}_S(a_1, b_1)$, $\mathcal{X}_S(a_2, b_2)$ and $\mathcal{X}_S(b_1 \oplus a_2, b_3)$ are affine subspaces, is satisfied in many practical cases. Indeed, when an S-box σ has differential uniformity at most 4, i.e., when 4 is the maximal value in the difference distribution table of σ , all sets $\mathcal{X}_\sigma(a, b)$ and $\mathcal{Y}_\sigma(a, b)$ are affine subspaces (see e.g., Lemma 2 in [DR07]). Therefore, the hypothesis is satisfied when S has an SPN structure based on a smaller differentially 4-uniform S-box σ : in this case, $\mathcal{X}_S(a, b)$ (resp. $\mathcal{Y}_S(a, b)$) corresponds to the Cartesian product of sets of the form $\mathcal{X}_\sigma(a, b)$ (resp. $\mathcal{Y}_\sigma(a, b)$).

An interesting observation deduced from the previous corollary is that, in all the previously mentioned situations, if the exact probability of a 3-round differential characteristic is non-zero, then it is greater than or equal to the usual estimate p_{ind} .

2.2 When S is differentially 4-uniform

There is a specific case where the factor λ between the two probabilities can be easily lower-bounded: when S itself is a function with differential uniformity at most 4.

Theorem 1. *Let S be a permutation of \mathbb{F}_2^n with differential uniformity at most 4. Let a_1, b_1, a_2, b_2, b_3 be five nonzero elements in \mathbb{F}_2^n with $b_1 \neq a_2$. Let*

$$p_{\text{exact}} = \Pr_{x,y \in \mathbb{F}_2^n} [S(S(x) \oplus y \oplus b_1 \oplus a_2) \oplus S(S(x) \oplus y) = b_3 \text{ and } S(x \oplus a_1) \oplus S(x) = b_1 \\ \text{and } S(y \oplus a_2) \oplus S(y) = b_2]$$

and

$$p_{\text{ind}} = \Pr_{x \in \mathbb{F}_2^n} [S(x \oplus a_1) \oplus S(x) = b_1] \times \Pr_{y \in \mathbb{F}_2^n} [S(y \oplus a_2) \oplus S(y) = b_2] \\ \times \Pr_{z \in \mathbb{F}_2^n} [S(z \oplus b_1 \oplus a_2) \oplus S(z) = b_3]$$

Then

$$p_{\text{exact}} = 2^\ell p_{\text{ind}} \text{ with } \max(0, n - 6) \leq \ell \leq n - 2.$$

Moreover, if all three differentials (a_1, b_1) , (a_2, b_2) , and $(b_1 \oplus a_2, b_3)$ have probability 2^{1-n} , then $\lambda \in \{0, 2^{n-2}\}$.

Proof. We know from Corollary 1 that $p_{\text{exact}} = 2^\ell p_{\text{ind}}$ with $\ell = n - \dim(V_1 + V_2 + V_3)$. Since $V_1 + V_2 + V_3$ is a subspace of \mathbb{F}_2^n , its dimension does not exceed n and is also smaller than the sum of the dimensions of the three subspaces. In particular, it is smaller than 6 because the S-box has differential uniformity at most 4. We derive that

$$\max(0, n - 6) \leq \ell \leq n - 2.$$

Moreover, when all three subspaces V_1, V_2 , and V_3 have dimension 1, then

$$V_1 + V_2 + V_3 = \langle b_1, a_2 \rangle.$$

It follows that, in this case,

$$\lambda \in \{0, 2^{n-2}\}.$$

In other words,

$$p_{\text{exact}} \in \{0, 2^{-2n+1}\}.$$

□

Most notably, when $n > 6$, the exact probability of the differential path can never be equal to the product of the probabilities of the three constituent transitions.

Example 1. Theorem 1 can be verified for instance when S is the AES S-box, which operates on \mathbb{F}_2^8 . Most differentials for the AES S-box have probability 2^{-7} . For such differential paths, we can check that $p_{\text{exact}} \in \{0, 2^{-15}\}$. For instance, for $(a_1, b_1) = (0\text{x}e5, 0\text{x}18)$, $(a_2, b_2) = (0\text{x}01, 0\text{x}ca)$, and $b_3 = 0\text{x}b3$, there are exactly two pairs $(x, y) \in \mathcal{X}_S(a_1, b_1) \times \mathcal{X}_S(a_2, b_2)$ such that $(S(x) \oplus y)$ satisfies the differential $(b_1 \oplus a_2, b_3)$. Then, the probability of the whole differential path is 2^{-15} while all three differentials have probability 2^{-7} , i.e., $\lambda = 2^{-15+21} = 2^6$. This factor varies when some of the involved differentials have probability 2^{-6} . For $(a_1, b_1) = (0\text{x}33, 0\text{x}0f)$, $(a_2, b_2) = (0\text{x}01, 0\text{x}1f)$ and $b_3 = 0\text{x}b8$, the probability of the whole differential path is again 2^{-15} , while the first differential has probability 2^{-7} and the other two have probability 2^{-6} . We then have $\lambda = 2^{-15+19} = 2^4$.

The highest possible value for p_{exact} is $2^{n-2} \times (2^{-(n-2)})^3 = 2^{-2n+4}$. We now give a simple necessary condition on a_2 and b_1 for obtaining differential paths with this maximal probability.

Proposition 2. *Let S be a permutation of \mathbb{F}_2^n with differential uniformity exactly 4. If there exist nonzero $a_1, b_1, a_2, b_2, b_3 \in \mathbb{F}_2^n$ with $a_2 \neq b_1$ such that $p_{\text{exact}} = 2^{-2n+4}$, then there exist x and y in \mathbb{F}_2^n such that the second-order derivatives of S and S^{-1} satisfy*

$$D_{a_2}D_{b_1}S(x) = 0 \text{ and } D_{a_2}D_{b_1}S^{-1}(y) = 0,$$

where $D_uD_vS(x) = S(x) \oplus S(x \oplus u) \oplus S(x \oplus v) \oplus S(x \oplus u \oplus v)$.

Most notably, if S is an involution, then there exists a pair (a_2, b_1) satisfying

$$D_{a_2}D_{b_1}S(x) = 0 \text{ and } D_{a_2}D_{b_1}S^{-1}(y) = 0.$$

Proof. From Theorem 1, we know that $\lambda \leq 2^{n-2}$. It follows that $p_{\text{exact}} = 2^{-2n+4}$ if and only if $\lambda = 2^{n-2}$ (i.e., if $\dim(V_1 + V_2 + V_3) = 2$) and all the three involved differentials have probability $2^{-(n-2)}$. Since $b_2 \in \mathbb{F}_2^n$ is such that the differential (a_2, b_2) has probability $2^{-(n-2)}$, there exists $x, v_2 \in \mathbb{F}_2^n$ with $v_2 \neq \{0, a_2\}$ such that $\mathcal{X}_S(a_2, b_2) = x + \langle a_2, v_2 \rangle$. This implies that

$$S(x) \oplus S(x \oplus a_2) = b_2 = S(x \oplus v_2) \oplus S(x \oplus v_2 \oplus a_2)$$

leading to

$$D_{a_2}D_{v_2}S(x) = 0.$$

Similarly, a_1 is such that $\mathcal{Y}_S(a_1, b_1) = y + \langle b_1, v_1 \rangle$ for some $y, v_1 \in \mathbb{F}_2^n$ with $v_1 \notin \{0, b_1\}$. We now use the fact that, for any permutation S , $\mathcal{Y}_S(a, b) = \mathcal{X}_{S^{-1}}(b, a)$. Indeed, $y \in \mathcal{Y}_S(a, b)$ if and only if $x = S^{-1}(y)$ satisfies

$$S(x \oplus a) \oplus S(x) = b.$$

Then, we have

$$S(S^{-1}(y) \oplus a) = y \oplus b$$

which is equivalent to

$$S^{-1}(y) \oplus a = S^{-1}(y \oplus b),$$

i.e., $y \in \mathcal{X}_{S^{-1}}(b, a)$. From the same arguments as for v_2 , we deduce that

$$D_{b_1}D_{v_1}S^{-1}(y) = 0.$$

But, since $\lambda = 2^{n-2}$, we know that

$$\dim(V_1 + V_2) = \dim\langle a_2, b_1, v_1, v_2 \rangle = 2.$$

It follows that $v_1 \in \{a_1, b_1 \oplus a_2\}$ and $v_2 \in \{b_1, b_1 \oplus a_2\}$. This implies that $D_{a_2}D_{b_1}S(x) = 0$ and $D_{a_2}D_{b_1}S^{-1}(y) = 0$.

It is well-known that there is no pair of nonzero distinct elements (a, b) such that D_aD_bS takes the value 0 if and only if S is APN (i.e., its differential uniformity equals 2) [Nyb94]. In our case, S is not APN, implying that such a pair (a, b) exists. When S is an involution, it also satisfies $D_aD_bS^{-1}(y) = 0$ for some y . \square

Example 2 (ROADRUNNER S-box). It is easy to check that, for the ROADRUNNER [BS15] S-box, there is no pair of nonzero distinct elements (a_2, b_1) such that both $D_{a_2}D_{b_1}S$ and $D_{a_2}D_{b_1}S^{-1}$ vanish at some points. We then deduce that $p_{\text{exact}} \leq 2^{-5}$. By examining all second-order derivatives of this S-box which take the value 0, we have searched for all $(a_1, b_1, a_2, b_2, b_3)$ such that all three differentials have probability 2^{-2} and lead to a differential path with overall probability 2^{-5} . We have found 136 such configurations. One example is

$$a_1 = 0x8, b_1 = 0x4, a_2 = 0x1, b_2 = 0x1, b_3 = 0x8.$$

Among these patterns, the only one which satisfies $a_1 = a_2 \oplus b_1$ and such that also the differentials (a_2, b_1) and (a_2, b_3) have probability 2^{-2} is the one we will use in the next section:

$$a_1 = 0xc, a_2 = 0xd \text{ and } b_1 = b_2 = b_3 = 0x1,$$

and the configuration obtained by inverting the roles of a_1 and a_2 .

Example 3 (Klein S-box). The Klein [GNL11] S-box is an involution over \mathbb{F}_2^4 . Then, there exist some pairs of nonzero distinct elements (a_2, b_1) such that both $D_{a_2}D_{b_1}S$ and $D_{a_2}D_{b_2}S^{-1}$ vanish at some points. For instance, $a_2 = 0x1$ and $b_1 = 0x2$ satisfy this property. For this S-box, the differential path defined by

$$a_1 = 0xd, b_1 = 0x2, a_2 = 0x1, b_2 = 0x3, \text{ and } b_3 = 0xe$$

has overall probability 2^{-4} . In other words, any pair of elements (x, y) satisfying the first two differentials also leads to some $(S(x) \oplus y)$ which satisfies the third one.

3 Application to RoadRunner

3.1 Description of RoadRunner

ROADRUNNER is a lightweight block cipher recently proposed by Baysal and Sahin [BS15]. It has a Feistel network structure with a 64-bit block size and it supports both 80 and 128-bit keys. In the 80-bit version, the number of rounds is 10, whereas in the 128-bit version the number of rounds is 12. Whitening keys (WK_0 and WK_1) are applied to the left half of the block in the first and last round. The general structure of ROADRUNNER is depicted in Figure 3.

Round Function. ROADRUNNER's round function, named F , takes as input a 32-bit block L_i , a 96-bit subkey K_i , and a 32-bit constant C_i . The constant C_i for round i is the 32-bit value $N_r - i$, where N_r is the total number of rounds of the cipher as defined above.

The round function in ROADRUNNER consists of three subsequent applications of SLK , which is composed of a substitution layer followed by a linear layer and a key addition layer. After three SLK layers a single substitution layer (S) is performed. In between the second and third SLK layer the constant C_i is added (cf. Figure 3).

Key Schedule. The key expansion of the 128-bit ROADRUNNER version chops the key up in four 32-bit words. The round keys are permutations of these words. Similarly, in the 80-bit version the key is split into five 16-bit words, and the key schedule is a permutation of six words. Table 2 lists the exact permutations for the round and whitening keys.

Substitution Layer. The substitution layer S consists of a parallel composition of the 4×4 -bit S-box of Table 3¹ to every 4-bit nibble of the block.

Linear Layer. The linear layer L applies the function $L' : \mathbb{F}_2^8 \mapsto \mathbb{F}_2^8$ to each individual byte of the block

$$L'(x) = x \oplus (x \lll 1) \oplus (x \lll 2).$$

This construction is known to be invertible in general for distinct rotation offsets [Riv11], and the designers of ROADRUNNER argue that this particular set of rotation offsets has good diffusion properties.

¹ This is the "optimal" S-box 13 in [UCI+11, Table 4].

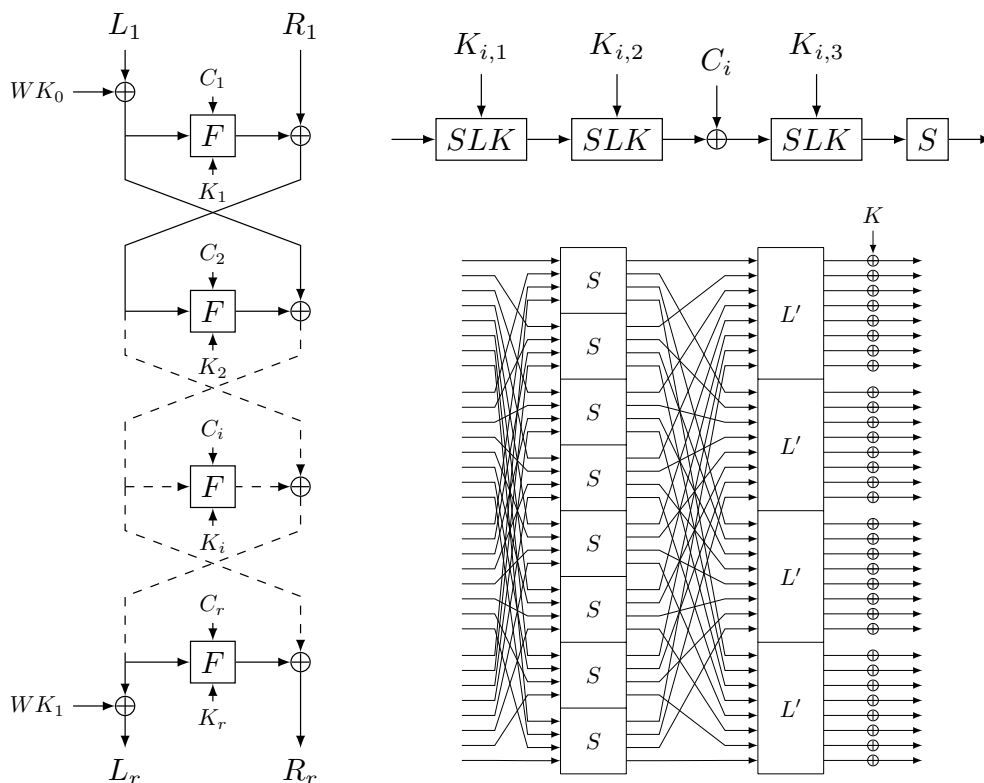


Figure 3: Overview of the ROADRUNNER block cipher. Left: Feistel network with whitening keys xored in the first and last round. Top right: The round function F , taking in as input a 32-bit word, a 32-bit constant and a 96-bit round key. Bottom right: The core SLK function, which consists of an S-box layer followed by a linear diffusion layer and finally a key addition.

3.2 Security Analysis by the Designers

The designers claim no security in the related-key setting, due to the fact that the key schedule uses the master key without any change in between rounds. The designers in fact mention in the paper that each F can be passed with only two active S-boxes in a related key attack, with total of 24 active S-boxes, and that this total number may be further reduced in a more detailed analysis. We stress that no information about concrete characteristics, such as plaintext and subkey difference is provided.

In the single-key setting, the designers show that the minimum number of active S-boxes in an active F is 10 along with concrete propagation patterns. The authors experimentally checked that the probability of characteristics and differentials is correct. In their experiments they report that, the differential probability does not significantly increase from the theoretically calculated characteristic probability. Based on this experiment, the authors assume that each active S-box multiplies the probability with 2^{-2} and an active F has approximately a probability of 2^{-20} .

3.3 Applications of our Observations

By comparing Figure 1 and Figure 3, it is easy to see that the analysis in Section 2 can directly be applied to ROADRUNNER when the number of rounds is more than two. We

Table 2: ROADRUNNER's key schedule.

(a) 128-bit key.		(b) 80-bit key.	
Round Number	Key schedule	Round Number	Key schedule
	WK_0		WK_0
	A		$A\ B$
	WK_1		WK_1
	B		$C\ D$
0 (mod 4)	$B\ C\ D$	0 (mod 5)	$C\ D\ E\ A\ B\ C$
1 (mod 4)	$A\ B\ C$	1 (mod 5)	$D\ E\ A\ B\ C\ D$
2 (mod 4)	$D\ A\ B$	2 (mod 5)	$E\ A\ B\ C\ D\ E$
3 (mod 4)	$C\ D\ A$	3 (mod 5)	$A\ B\ C\ D\ E\ A$
		4 (mod 5)	$B\ C\ D\ E\ A\ B$

Table 3: The ROADRUNNER S-box.

x	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
$S(x)$	0	8	6	D	5	F	7	C	4	E	2	3	9	1	B	A

emphasize that the observations can be applied both in the single-key and related-key settings. We also notice that the observation does not contradict the experiments by the designers that verified the probability of differentials within one round. What we are showing is that even before calculating the effect of collecting multiple differences, the actual probability of characteristics p_{exact} is higher than theoretically calculated one, p_{ind} , under the independent S-box assumption when the number of rounds is more than two.

In the following sections, we demonstrate the power of our observations with applications to concrete attacks.

3.4 Attack on RoadRunner-128

First, we concretize the characteristic having only two active S-boxes per round mentioned by the designers. Suppose that a 128-bit master key K is denoted by four 32-bit values and the difference of those values are denoted by $\Delta_0, \Delta_1, \Delta_2$ and Δ_3 . By following the key schedule described in Table 2, the difference of the initial whitening key is $\Delta WK_0 = \Delta_0$. Then, subkey differences are $(\Delta_1, \Delta_2, \Delta_3)$ for the first round, $(\Delta_0, \Delta_1, \Delta_2)$ for the second round, $(\Delta_3, \Delta_0, \Delta_1)$ for the third round, and so on. Four rounds with those subkey differences are illustrated in Figure 4.

We then choose $\Delta_0, \Delta_1, \Delta_2$ and Δ_3 . There are four S-layers in each round. Our strategy consists in canceling the difference from Δ_1 with Δ_2 after the S-layer, which makes the next S-layer inactive. Then canceling the difference from Δ_3 with Δ_0 after the S-layer, which makes the next S-layer inactive. By iterating this, non-active S-layers and active S-layers appear alternately, and we only have 2 active S-boxes per round.

As a result of our analysis, we construct a 4-round iterative characteristic by satisfying the following four conditions.

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_1) = \gamma_2] = 2^{-2}, \quad (1)$$

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_3) = \gamma_0] = 2^{-2}, \quad (2)$$

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_1) = \delta_1 \oplus \delta_3] = 2^{-2}, \quad (3)$$

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_3) = \delta_1 \oplus \delta_3] = 2^{-2}, \quad (4)$$

where δ_1 is a group of 4 bits in the 32-bit differences Δ_1 and the 4 bits gather into a single active S-box after the bit-permutation around the S-layer. δ_3 can similarly be defined.

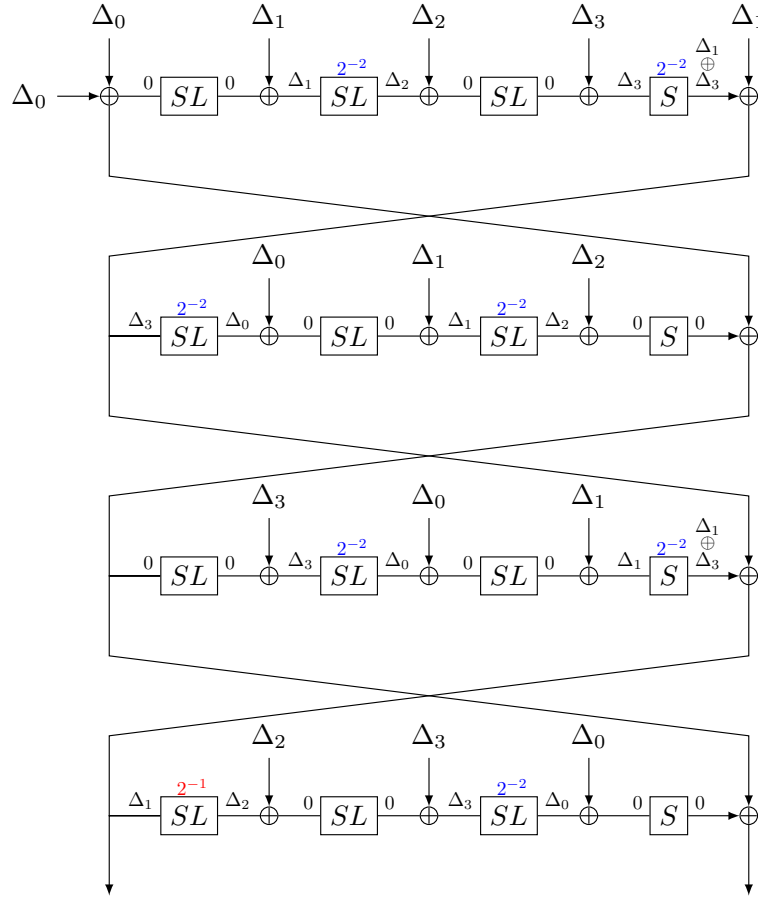


Figure 4: Four-round iterative differential characteristic against ROADRUNNER-128.

The difference γ_0 (resp. γ_2) corresponds to the corresponding nibble of $L^{-1}(\Delta_0)$ (resp. of $L^{-1}(\Delta_2)$) where L denotes the whole linear layer. For example, when the active S-box position is fixed to the top in Figure 3, $\delta_1 = 0xf$ corresponds to $\Delta_1 = 0x01010101$.

We note that by setting $\Delta_0 = \Delta_2 = L(\Delta_1 \oplus \Delta_3)$, the first two conditions can always be satisfied when the last two conditions are satisfied. The characteristic is iterative after 4 rounds including subkey differences, and can be extended to 12 rounds easily.

By analyzing the differential distribution table (DDT) of the S-box, we chose $\delta_1 = 0xc$ and $\delta_3 = 0xd$ (or $\Delta_1 = 0x01010000$ and $\Delta_3 = 0x01010001$). Then, $\delta_1 \oplus \delta_3 = 0x1$ ($\Delta_0 = \Delta_2 = L(0x00000001)$). This configuration satisfies the above listed conditions.

Evaluation of p_{ind} and p_{exact} . From Eqs. (1) to (4), p_{ind} can be calculated from the transition probability for each S-box, 2^{-2} , and the number of active S-boxes, leading to $2^{-2 \cdot 24} = 2^{-48}$.

Recall that for any pair (a, b) of differences, we use the following notation: $\mathcal{X}_S(a, b) = \{x \in \mathbb{F}_2^4 : S(x) \oplus S(x \oplus a) = b\}$ and $\mathcal{Y}_S(a, b) = \{S(x) \in \mathbb{F}_2^4 : S(x) \oplus S(x \oplus a) = b\}$. By applying the analysis in Section 2, p_{exact} of the first S-layer in round 4 in Figure 4 is

$$\Pr_{x, y \in \mathbb{F}_2^4} [x \oplus y \in \mathcal{X}_S(0xc, 0x1) | x \in \mathcal{X}_S(0xd, 0x1), y \in \mathcal{Y}_S(0xc, 0x1)]. \quad (5)$$

By analyzing DDT, we obtain $\mathcal{X}_S(0xd, 0x1) = \{0x0, 0x1, 0xc, 0xd\}$, $\mathcal{Y}_S(0xc, 0x1) = \{0x4, 0x5, 0xe, 0xf\}$, and $\mathcal{X}_S(0xc, 0x1) = \{0x4, 0x5, 0x8, 0x9\}$, which leads to $p_{\text{exact}} = 2^{-1}$.

Similarly, p_{exact} of the first S-layer in rounds 6, 8, 10, and 12 are 2^{-1} , which leads to 2^{-43} .

Experiments. First of all, we experimentally proved that 24 active S-boxes in 12 rounds is minimal by using the SAT-solver based tool [MP13]. Differently from the expectation by the designers, the number of active S-boxes will not be further reduced.

We then implemented the attack up to 8 rounds. We refer back to Table 1 for the results, which clearly indicates the gap between p_{ind} and p_{exact} in rounds 4, 6 and 8.

3.5 Attack on RoadRunner-80

In this part, we present an 8-round attack against ROADRUNNER-80. Differently from ROADRUNNER-128, the key is divided into 16-bit values (A, B, C, D, E) and each of them can be both the top half or the bottom half of 32-bit subkeys. Hence, constructing systematic subkeys is harder than in ROADRUNNER-128.

By applying the bit-permutation around S , a group of 4 bits for a single S-box will move to symmetric positions in the 32-bit state. To exploit this fact, we set $\Delta A = \Delta B = \Delta C = \Delta D = \Delta E$ to make all 32-bit subkey differences identical and symmetric.

We set subkey difference to the xor of two differences ΔX and ΔZ . ΔX takes a role of input difference to the subsequent S-layer, and ΔZ cancels the difference from the previous S-layer. Namely, in every S-layer, cancellation and injection of differences are performed. The characteristic is illustrated in Figure 5, which is iterative after four rounds.

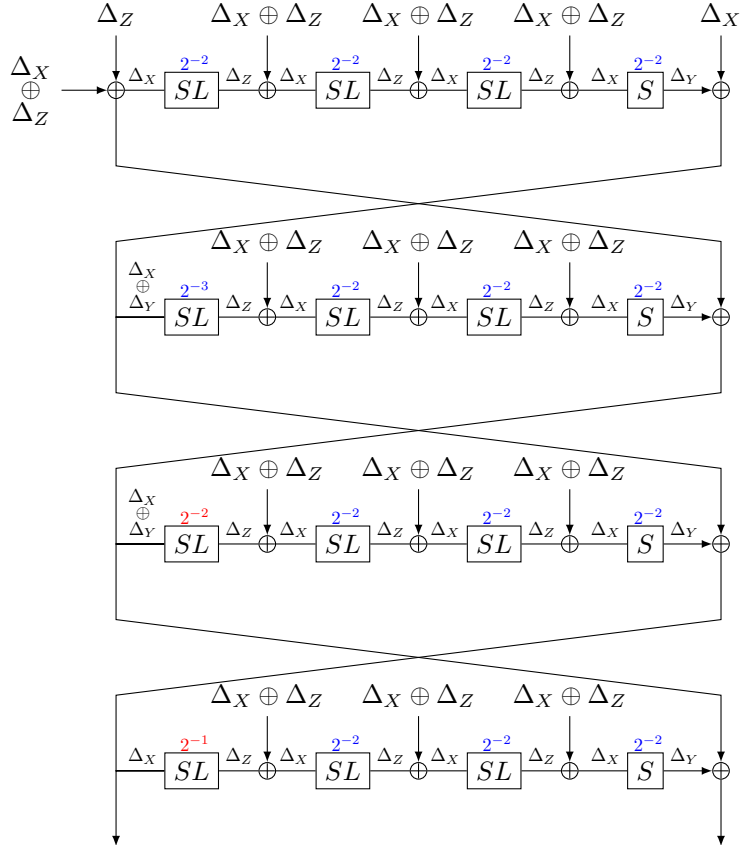


Figure 5: 4-round iterative characteristic for ROADRUNNER-80. $\Delta X = 0x5$, $\Delta Y = 0xA$, $\Delta Z = L(\Delta Y)$. The transition probabilities in red are those which differ from the estimate with the independent S-box assumption.

We then choose Δ_X and Δ_Z , where $\Delta_Z \triangleq L(\Delta_Y)$. We define δ_X, δ_Y similarly to the previous section, namely 4-bit difference in the 32-bit variable corresponding to an active S-box. Because subkey difference is symmetric, Δ_X and Δ_Y must be symmetric, which further limits δ_X, δ_Y to be symmetric (and non-zero). Therefore, $\delta_X, \delta_Y \in \{5, \mathbf{a}, \mathbf{f}\}$. According to the characteristic in Figure 5, we have the following two conditions;

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_X) = \delta_Y] > 0, \quad (6)$$

$$\Pr_{x \in \mathbb{F}_2^4} [S(x) \oplus S(x \oplus \delta_X \oplus \delta_Y) = \delta_Y] > 0, \quad (7)$$

$$\delta_X \neq \delta_Y. \quad (8)$$

From DDT, there is only one choice, $\delta_X = 5$ ($\Delta_X = 0\mathbf{x}00010001$) and $\delta_Y = \mathbf{a}$, which satisfies Conditions (6) and (7) with probability 2^{-2} and 2^{-3} , respectively.

Evaluation of p_{ind} and p_{exact} . We first evaluate p_{ind} . In every two rounds, there are seven active S-boxes with probability of 2^{-2} and there is one active S-box with probability of 2^{-3} . Thus p_{ind} is 2^{-17} in every 2 rounds and 2^{-68} for 8 rounds, which are unlikely to be satisfied with 2^{64} plaintexts of the full codebook.

The mechanism of occurring the advantage of p_{exact} is the same as in the attack against ROADRUNNER-128, but we now have an active S-box at the beginning of the inner function in every round. Therefore, from the third round, p_{exact} is higher than p_{ind} by a factor of 2, which improves the probability of 8-rounds to $2^{-8-9-8-7-7-8-8-7} = 2^{-62}$.

In more details, p_{exact} of the first S-layer in rounds with $p_{\text{ind}} = 2^{-8}$ and $p_{\text{ind}} = 2^{-9}$ are

$$\Pr_{x, y \in \mathbb{F}_2^4} [x \oplus y \in \mathcal{X}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a}) | x \in \mathcal{X}_S(0\mathbf{x}\mathbf{f}, 0\mathbf{x}\mathbf{a}), y \in \mathcal{Y}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a})], \quad (9)$$

$$\Pr_{x, y \in \mathbb{F}_2^4} [x \oplus y \in \mathcal{X}_S(0\mathbf{x}\mathbf{f}, 0\mathbf{x}\mathbf{a}) | x \in \mathcal{X}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a}), y \in \mathcal{Y}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a})]. \quad (10)$$

Given that $\mathcal{X}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a}) = \{0\mathbf{x}2, 0\mathbf{x}3, 0\mathbf{x}6, 0\mathbf{x}7\}$, $\mathcal{Y}_S(0\mathbf{x}5, 0\mathbf{x}\mathbf{a}) = \{0\mathbf{x}6, 0\mathbf{x}7, 0\mathbf{x}\mathbf{c}, 0\mathbf{x}\mathbf{d}\}$ and $\mathcal{X}_S(0\mathbf{x}\mathbf{f}, 0\mathbf{x}\mathbf{a}) = \{0\mathbf{x}0, 0\mathbf{x}\mathbf{f}\}$, p_{exact} in eq. (9) is 2^{-1} instead of 2^{-2} and p_{exact} in eq. (10) is 2^{-2} instead of 2^{-3} .

Experiments. To ensure our estimates match reality, we performed some computational verification of the above differential characteristic:

- 1 round of ROADRUNNER-80 yielded 65870 ($\approx 2^{16}$) matches over 2^{24} trials;
- 2 rounds of ROADRUNNER-80 yielded 1011 ($\approx 2^{10}$) matches over 2^{27} trials;
- 3 rounds of ROADRUNNER-80 yielded 124 ($\approx 2^7$) matches over 2^{32} trials;
- 4 rounds of ROADRUNNER-80 yielded 28 ($\approx 2^5$) matches over 2^{37} trials;
- 5 rounds of ROADRUNNER-80 yielded 16 ($= 2^4$) matches over 2^{43} trials.

These results are summarized in Table 1.

4 Extension to Almost-MDS Matrix in Minalpher-P

In this section, we show that improving the probability by evaluating p_{exact} can be extended to SPN with almost-MDS binary matrices. An example of such matrices is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad (11)$$

which is actually adopted by Minalpher [STA⁺14]. The rotated version of the above matrix is more popular, which can be seen in several designs e.g. PRINCE [BCG⁺12], FIDES [BBK⁺13], and Midori [BBI⁺15]. Section 4.1 provides an overview of our observation. Section 4.2 introduces the specification of Minalpher-P. Section 4.4 introduces the previous best differential characteristic evaluated by p_{ind} . Section 4.4 improves the probability by evaluating p_{exact} and extends the attack by two rounds.

4.1 Overview

Let us consider a 1-column state consisting of four cells of size n bits, thus the state size is $4n$ bits. Suppose that the state is updated by an SPN, in which the S-layer applies an n -bit S-box to all of four cells and the P-layer applies the matrix in Eq. (11). With this structure, the number of active cells can be two per rounds owing to the following property: *When two cells have an identical difference, the matrix multiplication does not change the number of active cells and the differential value.*

Let us consider the 2-round characteristic shown in Figure 6, which assumes that $\Pr_{x \in \mathbb{F}_2^n} [S(x) \oplus S(x \oplus \Delta a) = \Delta b] = 2^{-n+2}$ and $\Pr_{x \in \mathbb{F}_2^n} [S(x) \oplus S(x \oplus \Delta b) = \Delta c] = 2^{-n+2}$. p_{ind} is $(2^{-n+2})^4$ because of the four active S-boxes, meanwhile we show that p_{exact} is

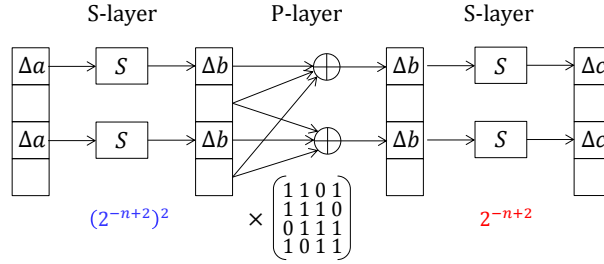


Figure 6: Overview: 2-round characteristic in SPN with single column.

$(2^{-n+2})^3$ in which the S-layer can be satisfied only with 2^{-n+2} from the second round. The state of SPN ciphers usually have more columns, thus the improvement by a factor of 2^{-n+2} can be amplified, which makes the improved factor significantly large.

4.2 Specification of Minalpher-P

The core part of Minalpher is the Even-Mansour construction in which a 256-bit plaintext is masked by a 256-bit secret value, and then a nibble-wise 256-bit permutation called Minalpher-P is computed. Finally, the output of Minalpher-P is masked by the 256-bit secret value. A 256-bit state is described as two 4×8 nibble-matrices denoted by A and B .

Let A_{i-1} and B_{i-1} be the inputs of the round function for round i . The states are updated to A_i and B_i with a round function, which consists of SubNibbles (SN), ShuffleRows (SR), SwapMatrices (SM), XorMatrix (XM) and MixColumns (MC), where SN , SR and MC are functions from $\{\mathbb{F}_2^4\}^{4 \times 8}$ to $\{\mathbb{F}_2^4\}^{4 \times 8}$. In the end, the state is xored with the round constant. We use notations A^{op} and B^{op} to denote $\{\mathbb{F}_2^4\}^{4 \times 8}$ data after operation op . See Figure 7 for its illustration.

SubNibbles (SN). SN substitutes each nibble by using 4-bit involution S-box S .

x	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
$S(x)$	B	3	4	1	2	8	C	F	5	D	E	0	6	9	A	7

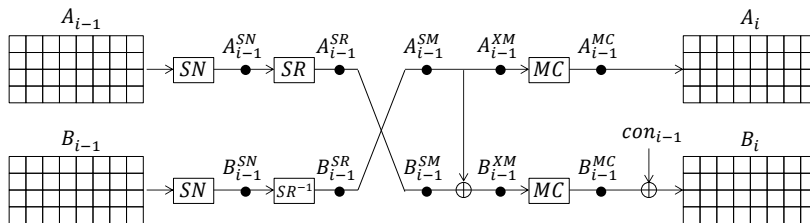


Figure 7: Illustration of the round function of Minalpher-P.

ShuffleRows (SR). SR shuffles nibble positions within each row. SR consists of two shuffle functions SR_1 and SR_2 defined as follows. Elements in 4×8 matrix A are moved according to the table below, and for B , SR^{-1} is applied instead of SR .

i	0	1	2	3	4	5	6	7
$SR_1(i)$	6	7	1	0	2	3	4	5
$SR_2(i)$	4	5	0	1	7	6	2	3
$SR_1^{-1}(i)$	3	2	4	5	6	7	0	1
$SR_2^{-1}(i)$	2	3	6	7	0	1	5	4

SwapMatrices (SM). SM swaps the matrix A and the matrix B .

XorMatrix (XM). The matrix B is xored with the matrix A .

MixColumns (MC). MC is a column-wise linear operation. As introduced before, MC is expressed as a multiplication by the matrix in Eq. (11).

Round Constant. The round constant con_{i-1} is xored to the matrix B . In this paper, the fact that the matrix A is not updated by round constant is important.

4.3 Differential Characteristics of Minalpher-P

The designers of Minalpher found a 6-round iterative truncated differential with 64 active S-boxes, which is shown in Figure 8. Note that this is not the one with minimal number of active S-boxes for 6 rounds. However, if it is iterated beyond 6 rounds, the number of active S-boxes matches the lower bound obtained by automated search.

Then, we convert the truncated differential to a specific characteristic by fixing the differential values. By calculating DDT of the 4-bit S-box, we observe that the input difference $0x4$ will be mapped to the output difference $0x4$ with probability 2^{-2} . So, we replace all filled cells in Figure 8 with the particular difference $0x4$.

Let us evaluate the probability of the 6-round characteristic. Here we assume that the secret mask of the Even-Mansour construction prevents the attacker from choosing the plaintext or ciphertext to deterministically satisfy differential propagations through S-box in the first and the last rounds. The linear part is satisfied with probability 1, thus the probability only comes from the S-box, which is 2^{-2} per S-box. Because $8 + 16 + 8 + 8 + 16 + 8 = 64$ S-boxes are included in the characteristic, the probability is $(2^{-2})^{64} = 2^{-128}$ when all transitions through all S-boxes are assumed to be independent. Considering that the security of Minalpher is claimed up to 128 bits, extending the characteristic by a few more rounds is impossible.

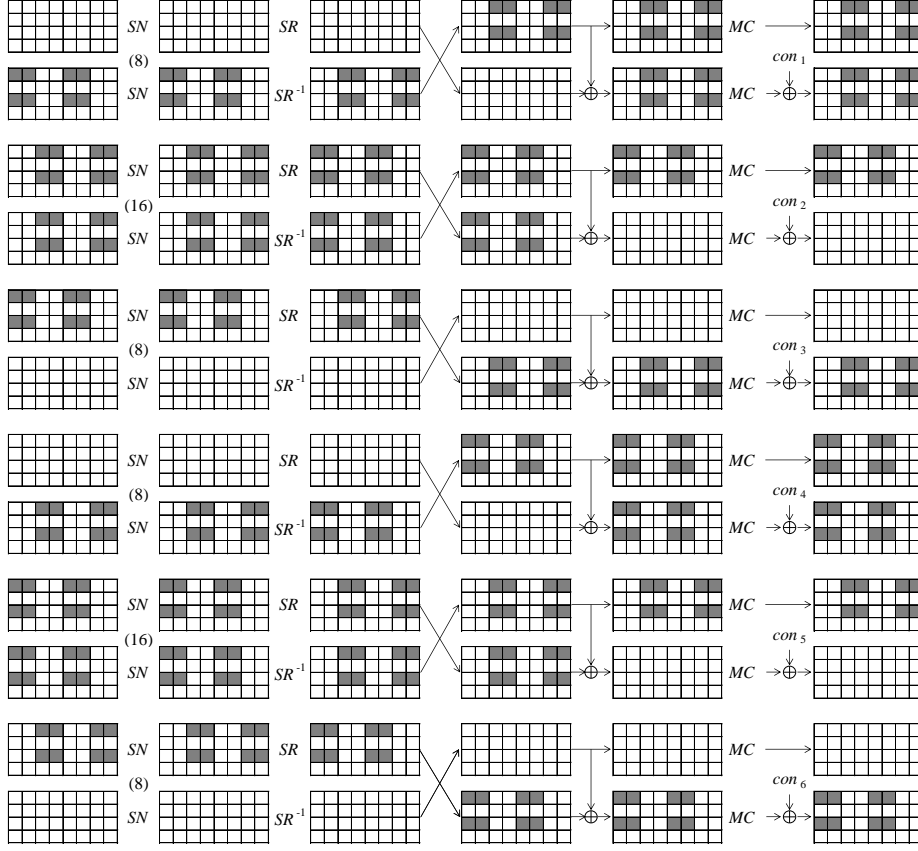


Figure 8: 6-round iterative truncated differential of Minalpher-P. Filled and empty cells denote active and inactive nibbles, respectively. Note that we rotated the original 6-round iterative characteristic by one round to optimize it in our analysis.

4.4 Analysis of Exact Probability

Preliminaries. Recall that for any pair (a, b) of differences, we use the following notation: $\mathcal{X}_S(a, b) = \{x \in \mathbb{F}_2^4 : S(x) \oplus S(x \oplus a) = b\}$ and $\mathcal{Y}_S(a, b) = \{S(x) \in \mathbb{F}_2^4 : S(x) \oplus S(x \oplus a) = b\}$. When S is involution as in Minalpher-P, $\mathcal{X}_S(a, a)$ is equal to $\mathcal{Y}_S(a, a)$ for any a . In particular, when $a = 4$ in the S-box of Minalpher-P, $\mathcal{X}_S(4, 4) = \mathcal{Y}_S(4, 4) = \{9, \mathbf{a}, \mathbf{d}, \mathbf{e}\}$. This is represented by an affine space $\langle 3, 4 \rangle + 9$, where $\langle x, y \rangle$ is a linear subspace.

Analysis of p_{exact} . Here, we show that the probability of the 6-round characteristic is actually 2^{-96} instead of 2^{-128} , thus the number of attacked rounds can be extended. We begin with the analysis of the simple case; SN and MC are iterated twice in a column, which is shown in Figure 9.

As shown in Figure 9, the five states are denoted by $X_1, X_1^{SN}, X_1^{MC}, X_2^{SN}, X_2^{MC}$. Suppose that the 4-nibble value of X_1 is chosen uniformly at random. Then the probability of satisfying the first SN layer is $(2^{-2})^2 = 2^{-4}$. When this occurs, the value of $X_1^{SN}[0]$ and $X_1^{SN}[2]$ are limited to four choices in $\mathcal{Y}_S(4, 4) = \{9, \mathbf{a}, \mathbf{d}, \mathbf{e}\}$. From the specification of

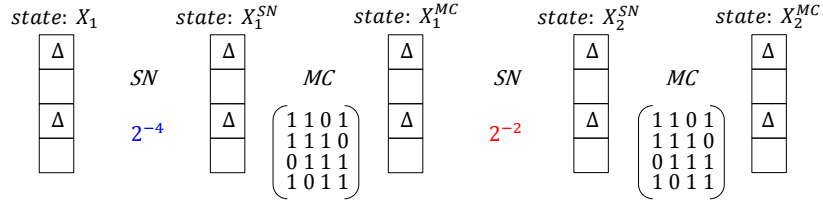


Figure 9: Analysis of simple case ($\Delta = 0x4$). Probability is 2^{-8} if two SN operations are evaluated independently, while the exact probability is 2^{-6} .

MC , the value of active nibbles in X_1^{MC} are calculated as

$$\begin{aligned} X_1^{MC}[0] &= X_1^{SN}[0] \oplus X_1^{SN}[1] \oplus X_1^{SN}[3], \\ X_1^{MC}[2] &= X_1^{SN}[1] \oplus X_1^{SN}[2] \oplus X_1^{SN}[3]. \end{aligned}$$

In order to satisfy the differential propagation in the second SN operation, both of $X_1^{MC}[0]$ and $X_1^{MC}[2]$ must be in the affine space of $\mathcal{X}_S(4, 4) = \{9, a, d, e\}$. Considering that $X_1^{SN}[0]$ and $X_1^{SN}[2]$ are in the affine space, the condition that both of $X_1^{MC}[0]$ and $X_1^{MC}[2]$ are in the same affine space is $X_1^{SN}[1] \oplus X_1^{SN}[3]$ is in its linear subspace $\langle 3, 4 \rangle = \{0, 3, 4, 7\}$. This occurs with probability 2^{-2} , thus the probability of satisfying the second SN layer is 2^{-2} , instead of 2^{-4} .

Application to 6-Round Characteristic. All the differences in Figure 8 are fixed to $0x4$.

Round 1. Suppose that the lower half of the input state, B_0 , is chosen uniformly at random. Then, the probability of satisfying the SN layer in round 1 is $(2^{-2})^8 = 2^{-16}$.

Round 2. The SR operation does not mix the value, thus irrelevant to this analysis. The state B_0^{SN} is next updated by MC and then passed to SN in round 2. Namely, the simple column-wise analysis discussed above appears in four columns. Thus the probability that the differences in A_1 are propagated to A_1^{SN} is $(2^{-2})^4 = 2^{-8}$ instead of 2^{-16} . Note that B_0^{SM} is xored with random state value B_0^M and round constant, thus the probability between B_1 and B_1^{SN} is 2^{-16} . In total, the probability of round 2 is 2^{-24} .

Round 3. The same event as round 2 occurs. Namely B_1^{SN} is updated with MC and then SN in round 3. As discussed before, this probability is 2^{-8} instead of 2^{-16} .

Rounds 4–6. The last three rounds are basically the iteration of the first three rounds in slightly different active-nibble positions. The probability evaluation is the same as one in Rounds 1–3, which is $2^{-16-24-8} = 2^{-48}$.

From the above discussion we conclude that the probability of the 6-round differential characteristic in Figure 8 is 2^{-96} , which is significantly larger than p_{ind} of 2^{-128} .

Experimental Verification. The probability of the first three rounds already reach 2^{64} , which is infeasible in our environment. The gap between p_{ind} and p_{exact} first appears in state A_1^{SB} of the SN operation in the second round, which is independent of the propagation in state B_1^{SB} . We thus implement the state update from B_0^{SB} to A_1^{SB} with the limitation that values of active bytes are sampled randomly from $\mathcal{Y}_S(4, 4)$.

We generated 65,536 ($= 2^{16}$) random values at B_0^{SB} , and 250 ($\approx 2^8$) values satisfy the difference in A_1^{SB} , which confirms that the probability of the characteristic from B_0^{SB} to A_1^{SN} is actually $(2^{-2})^4 = 2^{-8}$ instead of $(2^{-4})^4 = 2^{-16}$.

Extension to 8 Rounds. We append 1 round to both of the beginning and the end of the 6-round iterative characteristic in Figure 8. Remember that the probability of the first round in the 6-round characteristic is 2^{-16} . Due to the iterative structure, with the same reason, the probability of the last extended round is 2^{-16} . The extended round at the beginning has eight active S-boxes. Because the advantage of p_{exact} cannot be exploited at the beginning, the probability is $(2^{-2})^8 = 2^{-16}$.

To conclude, the probability of the 8-round characteristic is $2^{-96-16-16} = 2^{-128}$. Considering that the previous 6-round characteristic has the same probability, we improved the number of attacked rounds by 2 rounds.

5 Concluding Remarks

This paper studied the interaction between the differential transitions occurring in the multiple rounds of a fixed-key or unkeyed primitive. We showed that assuming independent input values for each S-box does not correspond to the actual situation, and p_{exact} can be much larger than p_{ind} . Our general analysis on the Feistel network showed that the gap between p_{exact} and p_{ind} depends on the S-box size and the S-box choice. In addition, having non-zero gap is inevitable when the S-box has differential uniformity 4 and a size larger than six bits.

This observation actually impacts the security of practical algorithms. We applied it to the lightweight block cipher ROADRUNNER and the authenticated encryption scheme Minalpher. The results showed that with p_{exact} the number of attacked rounds could be improved compared to the evaluation with p_{ind} .

Symmetric-key primitives with unkeyed functions or public permutations are getting more popular due to its lightweight property and can be seen in many contemporary structures such as the sponge and the Even-Mansour constructions. This paper alerts us that the resistance against differential cryptanalysis needs to be analyzed carefully.

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